

Helmholtz' Theory of Concordance

In 1857 Helmholtz gave a lecture to an audience in Bonn¹ in which he posed the following question:

It is an acknowledged fact that the numbers of vibrations of concordant tones bear to each other ratios expressible by small whole numbers. But why? What have the ratios of small whole numbers to do with concord?

Why indeed!

The simplest example of concord is the octave and two notes an octave apart have frequencies in the simplest possible ratio namely 2:1. In fact, it is often claimed that these two notes are in some sense 'the same' – just at a different pitch. But it is not easy to see why two notes an octave apart can reasonably be said to be 'the same' when they are obviously so far apart.

Likewise, the frequencies of notes a fifth apart involve a ratio of 3:2; notes a fourth apart a ratio of 4:3 etc. etc.

Helmholtz then, rightly, proceeds to point out that the notes produced by musical instruments such as violins and trumpets etc. all contain overtones or harmonics which are all whole number multiples of the frequency of the fundamental. He also points out that the overtones of percussion instruments such as bells are not harmonic (i.e. they are not necessarily whole numbers of the fundamental) and that this is the reason (he claims) why these instruments do not sound as pleasant as the one with harmonic overtones.

But there is a serious problem with this theory.

The ear is a very remarkable organ which basically consists of a long tube called the cochlea filled with sensitive hairs each tuned to a different frequency. When presented with a sound waveform it is able to instantly carry out what is known as a Fourier transform and basically what it sends to the brain is information about the frequency and intensity of all the overtones which make up the waveform. (Information about the phase of the overtones is largely ignored.) When the ear hears the sound of a trumpet playing a note whose fundamental frequency is f , hairs tuned to the frequencies f , $2f$, $3f$, $4f$ etc. are excited to a greater or lesser extent. If a violin plays the same note, the same pattern of hairs will be excited but the relative strengths will be different. In this way the brain can deduce not only what note the instrument is playing but also what instrument is playing it². If, however, the note is played by a bell which has the following sequence of overtones: f , $1.9f$, $2.7f$, $3.6f$ etc. a totally different set of hairs will be excited and obviously the brain will recognize that the note is being played by a completely different instrument. But why should the violin sound more pleasing than the bell? In fact, who is to say that the sound of a violin *is* more pleasing than that of a bell anyway? All that the brain knows is that the pattern of excited hairs is different. There is absolutely nothing in the structure of the ear which can tell the brain that the hair which is excited by a frequency of $2f$ is in any way related to the hair which is excited by the frequency f . It is my opinion that if I had been brought up in Bali surrounded every day by the sound of the gamelan, I would regard the sound of a bell as highly concordant and would possibly recoil in horror at the sound of a violin!

In Helmholtz' view

The art of the bell-founder consists precisely in giving bells such a form that the deeper

1 Reproduced in "Popular Scientific Lectures" published in translation by Longmans, Green, and Co in 1889 pp54 - 93

2 Actually there is a bit more to it than this. If you cut off the transients at the beginning of the note and just play the sustained section it can be surprisingly difficult to distinguish a violin from a trumpet.

and stronger partial tones shall be in harmony with the fundamental tone as otherwise the bell would be unmusical.

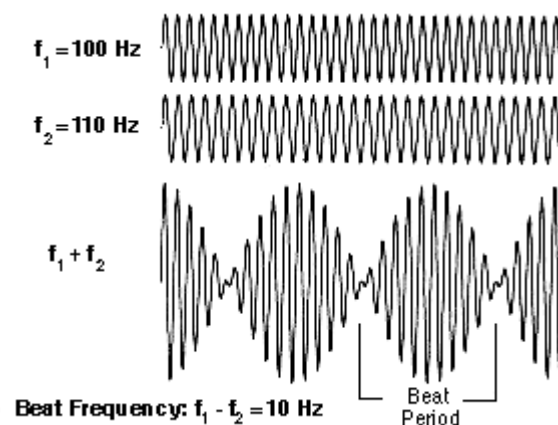
Here Helmholtz has committed the classic error of assuming the fact which he is trying to prove (that harmonic intervals are concordant). As we have seen, there is no *a priori* reason why harmonic intervals should necessarily be any more concordant than non-harmonic intervals.

We can, however, use Helmholtz' theory to explain why two notes an octave apart, played on an instrument rich in harmonics, sound 'the same'. The lower note will consist of the series of harmonics $f, 2f, 3f, 4f, 5f, 6f$ etc. while the upper note will consist of the harmonic series $2f, 4f, 6f$ etc. It is obvious that all the harmonics of the upper note are identical with some of the harmonics of the lower note. The brain detects this and interprets the notes as being, in some sense 'the same'.

Evidence that the untrained ear cannot instinctively hear an octave as being the same note is familiar to every primary school music teacher. Ask a young child to sing a note played on the piano and most will have no difficulty in getting somewhere close. Now ask them to sing the 'same note' but higher up and they will probably say that they haven't a clue what you mean! And if they do make a shot at it, it will probably be wildly out of tune. Hearing two notes an octave apart as being in some sense 'the same' is not an innate skill; it is learned.

Helmholtz then goes on to discuss another red herring – beats.

When two notes of slightly different frequency are sounded together – for example, when a note is struck on a badly tuned piano – the two waveforms move in and out of phase resulting in a waveform with a rapidly changing amplitude whose frequency is equal to the difference between the two original frequencies.



If the beat frequency is slow, this may manifest itself as a wow-wow sound, but when the beat frequency lies between 4 and 10 Hz, the sound becomes harsh and grating. The hairs in the cochlea which are excited are too close together to be distinguished by the brain, but the rapidly changing amplitude is detected. Only when the two frequencies are sufficiently far apart (about 15 Hz in the region of middle C) does the brain begin to perceive that two different notes are being played. Helmholtz then goes on to point out that if three notes of frequency $4f, 5f$ and $6f$ (e.g. middle C, E and G on the piano) are struck simultaneously, then each pair of adjacent notes will generate beat frequencies of f . It is the concordance of these 'secondary tones', Helmholtz claims, which is the reason why the major triad sounds so pleasing.

There are two difficulties with this theory. In the first place, it is very difficult to hear these very low 'concordant' frequencies. If the two notes sounded are middle C and E, the beat frequency is the C two octaves below middle C. There are people who claim that they can actually hear this note distinctly but the majority of people are quite unable to hear it. In fact, the beat frequency only becomes audible as a distinct note if there is significant non-linearity in the channel. Old fashioned AM radios used this effect. The audio information is coded onto a carrier wave of much higher

frequency by varying the amplitude of the signal. Alternatively, you could say that multiple carrier waves were being superimposed on each other generating beat frequencies in the audible spectrum. In order to retrieve the beat frequencies and reject the carrier wave, significant non-linearity is introduced into the circuit in the form of a diode. What this means is that it is only those people with defective hearing that can hear these 'secondary tones'! It seems very improbable that the pleasing sound of a major triad is in any way due to the concordance of these secondary tones.

The second objection to this theory is that the minor triad (whose frequencies are in the ratio 10:12:15) has secondary tones in the ratio of 2:3. What this means is that, if you could hear these tones they would not be coincident, they would be a fifth apart.

I think we must reject Helmholtz' theory of secondary tones outright.

So where does this leave us. At the very end of his lecture Helmholtz states:

...the determination of consonant intervals necessitates capabilities in the ear of feeling the upper partial tones and analysing the compound systems of waves into simple undulations according to Fourier's Theorem. It is entirely due to this theorem that the pitches of the upper partial tones of all serviceable musical tones must stand to the pitch of their fundamental tones in the ratios of the whole numbers to 1, and that consequently the ratios of pitches of concordant intervals must correspond with the smallest possible whole numbers.

Helmholtz is perfectly right in pointing out that the reason for the concordance of intervals to Western ears at least must be to do with the fact that the most common instruments used in classical western music are strings and pipes which have a rich set of harmonic overtones but Fourier's theorem has nothing to do with it. The reason why strings and pipes generate harmonic overtones (i.e. overtones whose frequencies are a whole number of the fundamental) is entirely physical and has nothing to do with Fourier's theorem. As I have said, bells generally have a strange set of overtones and they do not necessarily sound discordant to ears which have heard them from childhood.

There is, however, one further possibility which Helmholtz does not discuss. We have mentioned that when the beat frequency between two close notes lies between about 4 and 10 Hz the brain cannot correctly interpret the sound. Could it be that the reason why some intervals sound discordant be because there exist beat frequencies between the harmonics of the two notes?

When talking about the interval between two notes it is not very helpful to talk about frequencies because musical intervals are defined by ratios. It is much more useful to use a logarithmic scale such as the cent. One cent is defined as $1/1200^{\text{th}}$ part of an octave. The ratio which corresponds to 1 cent is therefore the 1200^{th} root of 2. This is equal to 1.00057779. This interval is far too small to detect. Since there are 12 semitones in an octave, the ratio which corresponds to 1. (equal temperament) semi-tone is the 12^{th} root of 2 or 1.059^3 . In other words 1 semi-tone = 100 cents (hence the name, I suppose). Few people can distinguish the difference between two notes less than 25 cents apart and two such notes sounded simultaneously sound discordant, presumably because the beat frequency generated is too rapid to follow⁴.

Here is a table showing the harmonics of a number of different notes above a fundamental in cents up to 5 octaves above the fundamental. Where a note shares⁵ a harmonic with that of the fundamental, that harmonic is shown in yellow. If the two harmonics clash⁶, the harmonic is shown in red. Note that the true harmonic intervals have been used rather than equal temperament intervals. (This is why the semitone, for example, is defined as 105 cents not 100 cents.)

3 To turn a ratio R into cents, use the formula $1200 \log(R)/\log(2)$

4 The A used to tune an orchestra has a frequency of 440 Hz. A note 25 cents higher has a frequency of 446 Hz, These notes sounded together will generate a beat frequency of 6 Hz.

5 Two harmonics are deemed to be shared if the notes differ by less than 6 cents.

6 Two harmonics are deemed to clash if they differ by between 6 and 25 cents inclusive.

	Octave	Fifth	Fourth	Major third	Minor sixth	Minor third	Major sixth	Tone	Semitone	Tritone	Mainjor third
1	2	3/2	4/3	5/4	8/5	6/5	5/3	9/8	17/16		
0	1200	702	498	386	814	316	884	204	105	600	351
1200	2400	1902	1698	1586	2014	1516	2084	1404	1305	1800	1551
1902	3102	2604	2400	2288	2716	2218	2786	2106	2007	2502	2253
2400	3600	3102	2898	2786	3214	2716	3284	2604	2505	3000	2751
2786	3986	3488	3284	3172	3600	3102	3670	2990	2891	3386	3137
3102	4302	3804	3600	3488	3916	3418	3986	3306	3207	3702	3453
3369	4569	4071	3867	3755	4183	3685	4253	3573	3474	3969	3720
3600	4800	4302	4098	3986	4414	3916	4484	3804	3705	4200	3951
3804	5004	4506	4302	4190	4618	4120	4688	4008	3909	4404	4155
3986	5186	4688	4484	4372	4800	4302	4870	4190	4091	4586	4337
4151	5351	4853	4649	4537	4965	4467	5035	4355	4256	4751	4502
4302	5502	5004	4800	4688	5116	4618	5186	4506	4407	4902	4653
4441	5641	5143	4939	4827	5255	4757	5325	4645	4546	5041	4792
4569	5769	5271	5067	4955	5383	4885	5453	4773	4674	5169	4920
4688	5888	5390	5186	5074	5502	5004	5572	4892	4793	5288	5039
4800	6000	5502	5298	5186	5614	5116	5684	5004	4905	5400	5151
4905		5607	5403	5291	5719	5221	5789	5109	5010	5505	5256
5004		5706	5502	5390	5818	5320	5888	5208	5109	5604	5355
5098		5800	5596	5484	5912	5414	5982	5302	5203	5698	5449
5186		5888	5684	5572	6000	5502		5390	5291	5786	5537
5271			5769	5657		5587		5475	5376	5871	5622
5351			5849	5737		5667		5555	5456	5951	5702
5428			5926	5814		5744		5632	5533		5779
5502			6000	5888		5818		5706	5607		5853
5573						5889			5678		5924
5641									5746		5992
5706									5811		6057
5769									5874		6120
5830									5935		
5888									5993		
5945									6050		
6000									6105		
6053											

We have already noted that the octave shares all of its harmonics with the fundamental. This makes the octave unique and must be the principal reason why two notes an octave apart 'sound the same'.

The fifth, fourth, major third and minor sixth all share harmonics to some degree and, crucially, none of their harmonics clash. The minor third and its counterpart the major sixth begin to exhibit some clashes between their harmonics. It is clear, however, from the large block of red which surrounds the shared harmonics exhibited by the semitone and the tritone that these intervals are going to be particularly discordant.

I think we now have our answer to the question which Helmholtz posed at the beginning of his lecture:

Two notes sounded by instruments rich in harmonics sound pleasing in respect of the degree to which their *harmonics are shared rather than clash*.

To test this theory, let us consider a note which is exactly half way between the fundamental and the fifth. i.e. 351 cents above the fundamental. A triad consisting of this note together with the fundamental and the perfect fifth would sound neither major nor minor and would, I think, sound particularly horrible. If so, we would expect the harmonics of this 'mainjor third' to have few, if any, shared harmonics with the other notes and many clashes. And this is exactly what we find. The last column contains no shared harmonics at all and is almost completely red above a certain frequency. (Clashes with the fifth are shown in dark red.) No wonder this note sounds so discordant to western ears.

I say 'western ears' because the theory only works if the instruments which are playing the notes are rich in harmonics. I would not expect a child brought up exclusively on the sound of bells and drums whose overtones are not harmonics of the fundamental to have the same notion of what

sounds concordant.

What happens if the instruments playing the notes have no overtones at all? A sine wave generator produces such a note. According to the theory suggested above, two notes sounded by two such generators should have no harmonics which clash and, provided that they are a semitone or more apart, should therefore sound concordant whatever the interval!

Well the first thing to say is that the bare fifth produced by two sine wave generators can hardly be described as harmonious. It just sounds like two unrelated notes and no more or less harmonious than any other pair of notes chosen at random. Even the 'mainjor third' sounds tolerable. (If you claim that the 'mainjor triad' played on sine wave generators still sounds horrible then I would suggest that is because you have *learned* to love the major and minor triads and not because your brain detects any particular discordance.)

It is also worth mentioning that, ever since equal temperament became popular, it is no longer even true to say that the frequencies of concordant intervals are in exact whole number ratios. They may be close but they are not exact⁷ and we have learned to like them.

In short, what Helmholtz has left out from his analysis of why certain intervals sound concordant and others less so is the role of the brain. Our brains have a huge capacity for adapting and learning. It is quite astonishing to think that a trained musician can hear a wrong note played by the second clarinet while the whole orchestra is playing – but this ability has to be learned. The ability to sing a tune in tune is not innate. It too has to be learned. A child surrounded by the sounds of strings and pipes will learn to find concord in notes which share a lot of harmonics. A child brought up in a culture full of the sounds of drums and bells will learn to hear sounds in a totally different way. The true answer to the question of why notes an octave apart 'sound the same' and why minor triads sound sad lies not in the physics of sound wave or the mathematics of Fourier Transforms – it lies in our brains.

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⁷ The principal notes of the equal temperament scale are all within 6 cents of the harmonic frequency except for the major third which is 14 cents too sharp and the minor third which is 16 cents too flat.